

Aharonov–Bohm Effect and Coordinate Transformations.

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Abstract

Resorting to a *Gedankenexperiment* which is very similar to the famous Aharonov–Bohm proposal it will be shown that, in the case of a Minkowskian spacetime, we may use a nonrelativistic quantum particle and a noninertial coordinate system and obtain geometric information of regions that are, to this particle, forbidden. This shows that the outcome of a nonrelativistic quantum process is determined not only by the features of geometry at those points at which the process takes place, but also by geometric parameters of regions in which the quantum system can not enter. From this fact we could claim that geometry at the quantum level plays a non–local role. Indeed, the measurement outputs of some nonrelativistic quantum experiments are determined not only by the geometry of the region in which the experiment takes place, but also by the geometric properties of spacetime volumes which are, in some way, forbidden in the experiment.

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1 Introduction.

Quantum Theory (QT) has become one of the most successful human achievements, and almost all of physics now relies upon QT. Nevertheless, there are some old conceptual puzzles that still beset this theory. For instance, the so called quantum measurement problem (the problem of the quantum limit) [1], the possible incompleteness of the general-relativistic description of gravity in the context of QT [2], or the possible discrepancy, in a curved manifold, between Feynman and Schrödinger formalisms [3].

At the classical level gravity can be understood as a purely geometric effect, the motion of a free classical particle moving in a curved manifold is given by the Weak Equivalence Principle (WEP), i.e., the particle moves along geodesics. The inclusion of additional interactions is done resorting to SEP, the famous “semicolon goes to coma rule” [4]. This principle tells us that locally the laws of physics are the special-relativistic laws.

Classically the role of geometry is local, i.e., the dynamics of a free classical particle located at a certain point P of any Riemannian manifold is, according to General Relativity (GR), determined by the geometric properties of this manifold at P (the motion equations can be written in terms of the connection coefficients, which at P depend only on the values of the components of the metric and their derivatives evaluated at P), geometry at any other point plays no role in the determination of the motion when the particle is at P . If we consider the geodesic deviation between two particles, then we would obtain information of Riemann tensor, but once again only of the region where the motion of these classical particles takes place.

Nevertheless, at the quantum level the situation could be not so satisfactory. Indeed, the experiment of Colella, Overhauser and Werner [5] tells us that at the quantum level gravity is not anymore a purely geometric effect [6], the mass of the employed particles appears explicitly in the interference term. This fact emerges once again if we measure continuously the position of the particles [7] (even if the particles follow the same trajectory), and could lead to the emergence, in some cases, of something like a gravitational quantum Zeno effect [8].

In order to understand better the possible appearance of nonlocal effects in QT let us at this point address an already known similarity between electrodynamics and gravitation.

At the classical level the motion of a charged particle is solely determined by the force law of Lorentz and Newton’s second law. We also know that the electric and magnetic fields are invariant under the so called gauge transformations [9].

Nevertheless, this is not the situation at the quantum level. The role that the

concept of potential plays in physics has been deeply modified by Aharonov–Bohm effect (AB) [10]. Indeed, even though Lorentz force vanishes at those points at which the wave function has nonvanishing values, the dynamical behavior is sensitive to the existence of a magnetic field inside a region where the charged particle can never enter, the vector potential \mathbf{A} has in this effect a measurable consequence, detectable in the interference pattern of a charged particle. AB shows us also that in QT there are nonlocal effects, i.e., the features of the vector potential at points where the wave function vanishes affects the dynamics of the particle, in other words, we could say that in QT \mathbf{A} , sometimes, has a nonlocal role. This does not happen in the classical case, where forces have a local character. This effect has already been confirmed experimentally [11].

In classical physics \mathbf{A} is a gauge field, it has no physical meaning (at least before a gauge is imposed), it is the field strength which is physically relevant. Concerning geometry something similar happens in relation with the components of the metric tensor, $g_{\mu\nu}$. They are deprived of physical meaning (of course, the metric tensor has physically relevant meaning, but not its components), and sometimes play also the role of a potential, i.e., gravitational potential. Therefore we may wonder if we could find a construction in which (in analogy with the electrodynamical case, where a nongauge invariant field renders nonlocal effects in QT) these noninvariant (under coordinate transformations) parameters, $g_{\mu\nu}$, could allow us to find nonlocal effects in QT. It is in this sense that here we will speak of a coordinate transformations–induced Aharonov–Bohm effect, the appearance of a nonlocal behavior in nonrelativistic QT by means of coordinate transformations.

In this work we will consider a Minkowskian spacetime and two coordinate systems in it. One of them is an inertial system, while the second one is accelerated. We will prove resorting to a *Gedankenexperiment*, which is very similar to the AB case, that in the accelerated system nonlocal effects could appear in the context of nonrelativistic QT, and that this is a geometry–induced feature.

2 Transformations and Aharonov–Bohm Effect.

Consider a Minkowskian spacetime, and let us denote the coordinates of an inertial coordinate system by $x^{\tilde{\alpha}}$. The matrix elements of the coordinate transformation leading to a second coordinate system (which in general is noninertial, and whose coordinates will be denoted by x^{β}) are given by $\Lambda_{\tilde{\alpha}}^{\beta}$. In other words, we have $x^{\beta} = \Lambda_{\tilde{\alpha}}^{\beta} x^{\tilde{\alpha}}$. Notice that no conditions have been imposed upon $\Lambda_{\tilde{\alpha}}^{\beta}$.

Let us now proceed to analyze the movement of a quantum particle, and denote its corresponding Lagrangian by L . We will restrict ourselves to the case of low velocities, i.e., velocities much smaller than the speed of light.

The motion of a free classical particle in a Riemannian manifold is given by the corresponding geodesics [4]. In the case of a Minkowskian spacetime, an inertial coordinate system obtains the motion equations calculating the extremal curves of the following expression

$$S = \int (-\eta_{\tilde{\mu}\tilde{\nu}} \frac{dx^{\tilde{\mu}}}{d\tau} \frac{dx^{\tilde{\nu}}}{d\tau})^{1/2} d\tau. \quad (1)$$

But in the noninertial system one has to consider not expression (1) but

$$S = \int (-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau})^{1/2} d\tau. \quad (2)$$

From (2) we deduce the motion equations in the non-inertial system

$$\frac{d^2 x^\beta}{d\tau^2} + \Gamma_{\mu\nu}^\beta \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0, \quad (3)$$

here $\Gamma_{\mu\nu}^\beta$ are the so called connection coefficients, and τ represents proper time.

In other words, we may interpret expression (2) as the action of a classical particle in the noninertial system, and $L = (-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau})^{1/2}$ as its Lagrangian.

We now introduce in this Minkowskian spacetime a very particular coordinate system.

Consider a cylindrical volume H (this volume is infinitely long, and its cross section, denoted by E , has radius ρ_a , see figure). Let us now introduce a vector field, denoted by \mathbf{A} , such that it satisfies the following conditions: (i) inside H (if $0 \leq \rho \leq \rho_a$) we have, in cylindrical coordinates, $A_\rho = 0$, $A_z = 0$, and $A_\phi = F\rho/2$; (ii) outside H (if $\rho_a \leq \rho$), $A_\rho = 0$, $A_z = 0$, and finally $A_\phi = F\rho_a/2$. Here $\rho_a > 0$ is a fixed number and F is a nonvanishing real number. Clearly, \mathbf{A} is everywhere continuous. From this definition we may evaluate its rotational; $\nabla \times \mathbf{A} = \mathbf{0}$, if $\rho_a \leq \rho$, and $\nabla \times \mathbf{A} = F\hat{\mathbf{z}} = \mathbf{F}$, if $0 \leq \rho \leq \rho_a$, being $\hat{\mathbf{z}}$ the unit vector along the axis of symmetry.

We may now define, using vector field \mathbf{A} , the components of the metric of our accelerated coordinate system, namely $g_{0\phi,\rho} = A_\phi$, $g_{0z,\rho} = A_z$, $g_{0\rho,\rho} = A_\rho$, $g_{00,\nu} = 0$, and $g_{0l,j} = 0$, if $j \neq \rho$ (here l and j represent space coordinates, while ν denotes spacetime ones).

The mathematical consistency of this noninertial metric is determined by the existence of a coordinate transformation that could render the aforesaid conditions. In order to see that we may have this kind of coordinate system, we must note that in this situation we must determine 16 functions, namely $\Lambda_{\mu}^{\tilde{\beta}}$ (because we have differential equations in terms of the components of the metric of the accelerated system, and we also know that $g_{\mu\nu} = \Lambda_{\mu}^{\tilde{\beta}} \Lambda_{\nu}^{\tilde{\alpha}} \eta_{\tilde{\beta}\tilde{\alpha}}$). But we have only 13 equations, and in consequence the system, in principle, is solvable.

We now construct a *Gedankenexperiment* that could be considered, in some way, an extrapolation of the famous Aharonov–Bohm construction [10].

Take two points P (source point) and Q (detection point) in this manifold such that the above mentioned cylindrical volume lies between them (see figure).

A particle will move from P to Q . It first passes through a conventional two-slit device (here we consider each slit as a finite “hole”) [12], and afterwards enters a region in which a forbidden volume D for this particle exists (this volume D is infinitely long, and contains in its interior the, also infinitely long, cylinder H in which $\nabla \times \mathbf{A} \neq \mathbf{0}$). Then it is detected at point Q . In other words, after passing the two-slit device it remains always on one “side” of space, either “right” or “left”, volume D acts as a barrier for the particle.

Under these conditions the proper time of any curve C joining P and Q is given by

$$S = \int_C (-g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau})^{1/2} d\tau. \quad (4)$$

Let us now suppose that the source and detection points are moved along the ρ coordinate a distance $-\epsilon$, i.e., any curve $x^{\alpha} = C^{\alpha}(\tau)$ joining P and Q will become now $x^{\alpha} = C^{\alpha}(\tau)$ for $\alpha \neq \rho$, and $\rho(\tau) = C^{\rho}(\tau) - \epsilon$, with the condition $|\epsilon| \ll 1$.

In this new situation expression (4) becomes

$$S = \int_C (-g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} + \epsilon \frac{\partial g_{\mu\nu}}{\partial \rho} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau})^{1/2} d\tau. \quad (5)$$

We now consider a quantum particle moving (here only the case of low velocities is analyzed) from the new source point to the new detection point. The description of the movement of this particle can be done using Feynman’s path integral formulation for a nonrelativistic particle [13], thus its propagator U is given by

$$U(\mathbf{x}_2, \tau''; \mathbf{x}_1, \tau') = \int d[\mathbf{x}(\tau)] \exp\left(\frac{i}{\hbar} \int_{\tau'}^{\tau''} (-g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} + \epsilon \frac{\partial g_{\mu\nu}}{\partial \rho} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau})^{1/2} d\tau\right), \quad (6)$$

being \mathbf{x}_2 and \mathbf{x}_1 the space coordinates of the new detection point and of the new source point, respectively.

But low velocities ($\frac{dt}{d\tau} \sim 1$, here $c = 1$) and $g_{00,\rho} = 0$ imply that the propagator is approximately

$$U(\mathbf{x}_2, \tau''; \mathbf{x}_1, \tau') = \int d[\mathbf{x}(t)] \exp\left(\frac{i}{\hbar} \int_{\tau'}^{\tau''} (L + \epsilon A_l \frac{dx^l}{dt}) dt\right), \quad (7)$$

being $L = (-g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt})^{1/2}$.

3 Interference Terms.

Let us now calculate the probability of detecting our particle. Clearly, the propagator at this point is the sum of two terms, the propagator “right” and the propagator “left”.

$$U(\mathbf{x}_2, \tau''; \mathbf{x}_1, \tau') = \int_{(right)} d[\mathbf{x}(t)] \exp\left(\frac{i}{\hbar} \int_{\tau'}^{\tau''} [L + \epsilon A_l \frac{dx^l}{dt}] dt\right) + \int_{(left)} d[\mathbf{x}(t)] \exp\left(\frac{i}{\hbar} \int_{\tau'}^{\tau''} [L + \epsilon A_l \frac{dx^l}{dt}] dt\right). \quad (8)$$

As was mentioned before, in this *Gedankenexperiment* the particles can not go from the left-hand side to the right-hand side (or from the right-hand side to the left-hand side). These two conditions imply that in our case orbits are confined to a topologically restricted part of space.

If we carry out the so called “skeletonization” , then we may see that the contribution to the respective integrals of each trajectory can be written as follows

$$\prod_{n=1}^{N-1} \exp\left(\frac{i}{\hbar} S[n, n+1] + \frac{i}{\hbar} T_{top}\right) \exp\left(\epsilon \frac{i}{\hbar} \int_C \mathbf{A} \cdot d\mathbf{s}\right), \quad (9)$$

being C the trajectory under consideration joining new source point and new detection point, $s^l = \frac{dx^l}{dt} dt$, and S the action associated to L . The new term T_{top} is a pure boundary term, which keeps track of the imposed topological restrictions [14]. Either “right” or “left” the rotational of \mathbf{A} vanishes, therefore the line integral on the last term of expression (9) depends only on the initial and final points, and not on C (it

is readily seen that C is not a closed curve, and that it lies outside cylinder H). In other words, if we consider two trajectories “right” (“left”) the contribution of our vector field \mathbf{A} to each one of them is the same, i.e., the exponential of the line integral of \mathbf{A} is a common factor.

Hence the propagator becomes now

$$U(\mathbf{x}_2, \tau''; \mathbf{x}_1, \tau') = \exp\left(\epsilon \frac{i}{\hbar} \int_{C1} \mathbf{A} \cdot d\mathbf{s}\right) \int_{(right)} d[\mathbf{x}(t)] \exp\left(\frac{i}{\hbar} \int_{\tau'}^{\tau''} \tilde{L} dt\right) \\ + \exp\left(\epsilon \frac{i}{\hbar} \int_{C2} \mathbf{A} \cdot d\mathbf{s}\right) \int_{(left)} d[\mathbf{x}(t)] \exp\left(\frac{i}{\hbar} \int_{\tau'}^{\tau''} \tilde{L} dt\right). \quad (10)$$

Here $C1$ and $C2$ are any trajectory (joining points P and Q) “right” and “left”, respectively, and \tilde{L} represents the Lagrangian function L plus the topological term T_{top} .

From expression (10) we may evaluate the interference term.

$$I = 2\alpha \cos\left(\epsilon \frac{1}{\hbar} \oint_{\tilde{C}} \mathbf{A} \cdot d\mathbf{s}\right) - 2\beta \sin\left(\epsilon \frac{1}{\hbar} \oint_{\tilde{C}} \mathbf{A} \cdot d\mathbf{s}\right). \quad (11)$$

In expression (11) the closed curve \tilde{C} is defined with $C1$ and $C2$. Firstly, we move from the new source point to the new detection point along $C1$, and then backwards along $C2$. We have also introduced the following definitions

$$\alpha = \text{Re}\left\{ \int_{(right)} d[\mathbf{x}(t)] \exp\left(\frac{i}{\hbar} \int_{\tau'}^{\tau''} \tilde{L} dt\right) \int_{(left)} d[\mathbf{x}(t)] \exp\left(-\frac{i}{\hbar} \int_{\tau'}^{\tau''} \tilde{L} dt\right) \right\}, \quad (12)$$

and

$$\beta = \text{Im}\left\{ \int_{(right)} d[\mathbf{x}(t)] \exp\left(\frac{i}{\hbar} \int_{\tau'}^{\tau''} \tilde{L} dt\right) \int_{(left)} d[\mathbf{x}(t)] \exp\left(-\frac{i}{\hbar} \int_{\tau'}^{\tau''} \tilde{L} dt\right) \right\}. \quad (13)$$

Using Stokes’ theorem we may rewrite this interference term as

$$I = 2\alpha \cos\left(\epsilon \frac{1}{\hbar} \int_{\Omega} \nabla \times \mathbf{A} \cdot d\mathbf{\Omega}\right) - 2\beta \sin\left(\epsilon \frac{1}{\hbar} \int_{\Omega} \nabla \times \mathbf{A} \cdot d\mathbf{\Omega}\right), \quad (14)$$

being Ω an area bounded by \tilde{C} . But we have defined our vector field \mathbf{A} such that its rotational vanishes everywhere on Ω but in a small area located inside the forbidden volume D , i.e., in the cross section of cylinder H , which was denoted by E and has radius ρ_a . Hence the nonvanishing part of the line integrals allows us to rewrite (14) as follows

$$I = 2\alpha \cos\left(\epsilon \frac{1}{\hbar} \int_E \nabla \times \mathbf{A} \cdot d\mathbf{E}\right) - 2\beta \sin\left(\epsilon \frac{1}{\hbar} \int_E \nabla \times \mathbf{A} \cdot d\mathbf{E}\right). \quad (15)$$

From our previous definitions we obtain the final form of this interference term

$$I = 2\alpha \cos\left(\epsilon \frac{F}{\hbar} \pi \rho_a^2\right) - 2\beta \sin\left(\epsilon \frac{F}{\hbar} \pi \rho_a^2\right). \quad (16)$$

4 Discussion.

We have proved, using a *Gedankenexperiment* which is very similar to the famous Aharonov–Bohm proposal, that we may find noninertial coordinate systems (in a Minkowskian spacetime), in which a nonrelativistic quantum process is determined not only by the features of geometry at those points at which the process takes place, but also by geometric parameters of regions in which the quantum system can not enter.

This is a purely quantum mechanical effect. Indeed, if we had used a classical particle, where the motion at any point P of its trajectory is solely determined by the geometry at P , then no information of any forbidden region could be extracted.

The here introduced *Gedankenexperiment* could hardly be considered as a local experiment, nevertheless, from our results we could claim that geometry–induced nonlocal effects could emerge in QT. Indeed, the measurement outputs of some nonrelativistic quantum experiments are determined not only by the geometry of the region in which the experiment takes place, but also by the geometry of regions forbidden, in some way, to the experiment.

The gravitational field can be geometrized (at least at the classical level), and at this point we may wonder if at quantum level gravity could render some nonlocal effects in nonrelativistic QT (of course, the present work does not consider gravity, but

it has proved that in a Minkowskian spacetime there could be a geometry-induced nonlocality, and therefore the extension to curved spacetimes is an interesting question). This nonlocal behavior has already been pointed out [15], and more investigation around this topic could lead to a more profound comprehension of the way in which the gravitational field could modify some fundamental expressions of QT, for example the commutation relations [16, 17].

The possible incompleteness of the general relativistic description of gravity, at quantum level, has already been claimed [2], and implies the violation not only of Einstein equivalence principle but also of the local position invariance principle [18] (the independence of the results of local experiments from the location of the local laboratory in spacetime, i.e., the independence of the equivalence principle from position in time and space). In other words, Ahluwalia's work implies that the results of some local quantum experiments do depend on nonlocal characteristics. In the present work we have found a behavior that, at least qualitatively, is the same, i.e., the dynamics of some quantum processes is determined by nonlocal features. Hence, further investigation in this Aharonov-Bohm effect could help to understand better the controversy around the validity, at quantum level, of the equivalence principle

In the present work, the physical acceptability of the constructed noninertial system has not been investigated. In spite of this last fact, our work has shown that, in principle, there are noninertial systems, in which nonrelativistic QT shows very interesting nonlocal features. The possibility of having also these kind of effects in the context of more realistic schemes (feasible noninertial observers) has to be investigated, but at least we have shown that in a very wide range of noninertial coordinate systems these effects do exist.

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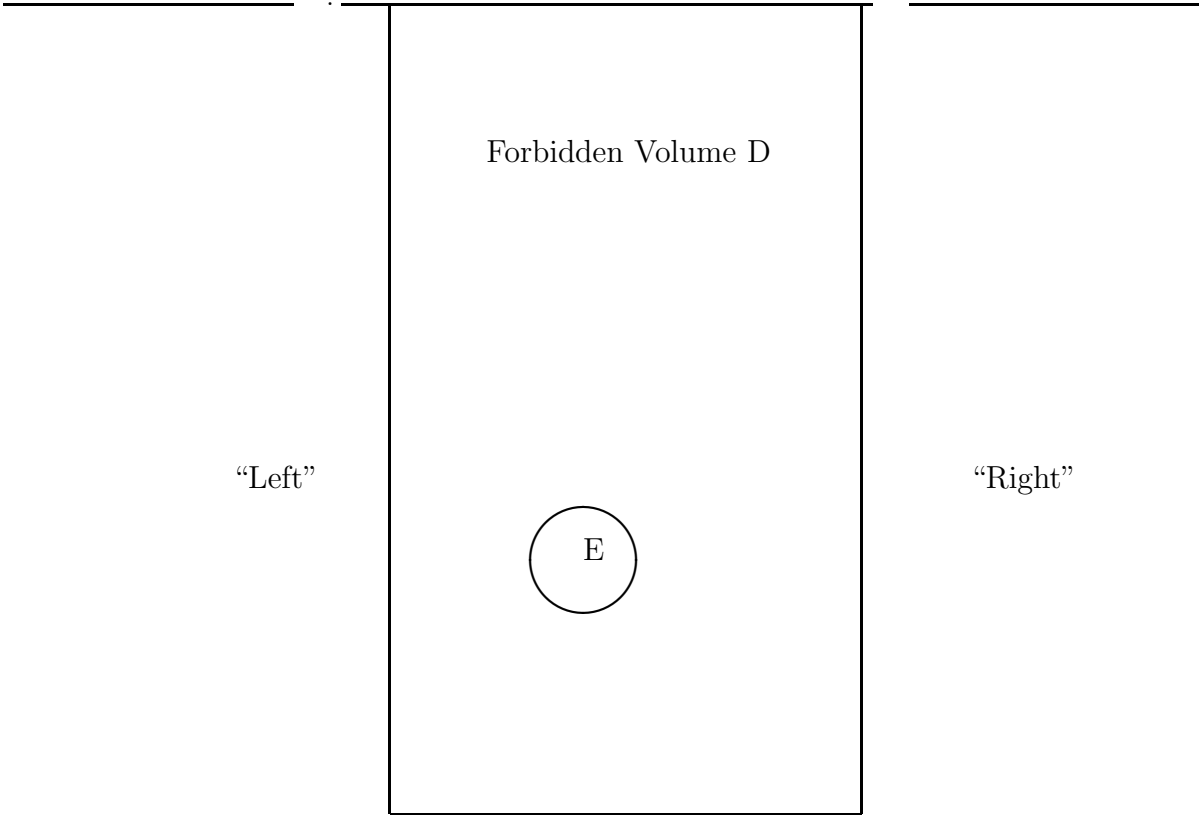
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Source Point.

P



Q
Detection Point.